A Highly Structured Euler Characteristic

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University of Southampton PGR Seminar

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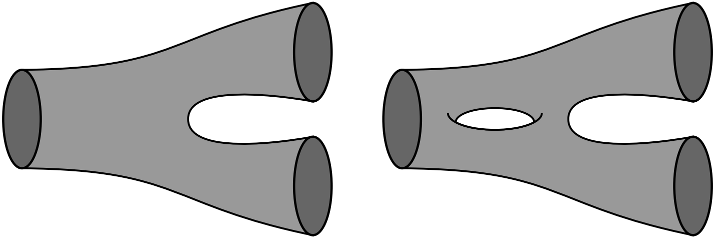

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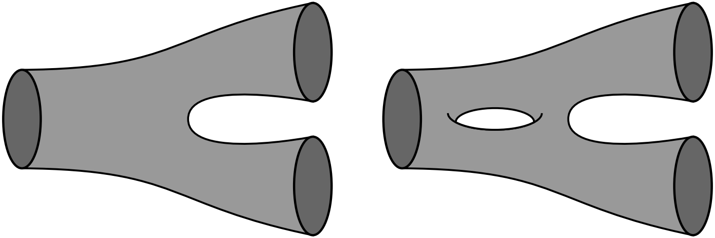

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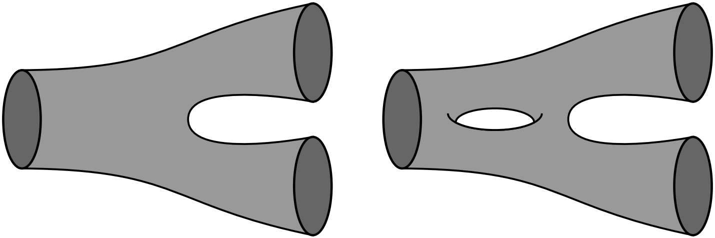

Figure: Two bordisms from S^1 to $S^1 \sqcup S^1$.

If there is a bordism from M to N, we say that M and N are *bordant*.
This defines an equivalence relation on manifolds. We write Ω_n for the set of bordism classes of n-manifolds, and $[M]$ for the bordism class of M.

The Bordism Ring

The set Ω_n admits an abelian group structure, where

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We can turn Ω_* into a graded ring, where

$$[M] \cdot [N] = [M \times N],$$

and the identity element is $[\mathrm{pt}]$.

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The assignments $X \mapsto \Omega_n(X)$ define a *generalised homology theory*. That is, the $\Omega_n(\text{pt})$ is nontrivial.

The Bordism Groups of a Point

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Algebraically, we have the following theorem.

Theorem (Thom, 1954)

There is an isomorphism of graded rings

$$\Omega_*(\mathrm{pt}) \cong \mathbb{F}_2[x_2, x_4, x_5, x_6, x_8, \ldots],$$

with an indeterminate in each positive degree n where $n \neq 2^i - 1$.

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There is a spectrum MO called the *unoriented bordism spectrum*. It represents the bordism homology theory.

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We can also define homotopy groups $\pi_n(X)$ of a spectrum X to be homotopy classes of maps from the *sphere spectrum* $\mathbb{S}^n$ into X. There are isomorphisms of abelian groups

$$\Omega_n(\mathrm{pt}) \cong \pi_n(MO).$$

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Thus we can define a *ring spectrum* to be a spectrum R together with a map $R \wedge R \to R$, which satisfies some identities in the homotopy category (analogous to those for rings).

Ring Spectra cont.

The spectra representing ordinary homology with coefficients in a ring R are called *Eilenberg-Mac Lane spectra*, and are denoted HR. These can be made into ring spectra.

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We know the structure of $\pi_*(MO)$ as a ring, but what about the structure of MO as a ring spectrum?

Ring Spectra of Characteristic 2

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Theorem (Boardman, 1980)

If R is a commutative ring spectrum of characteristic 2 and there is an additive equivalence

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Theorem (Pajitnov-Rudyak 1985)/(Würgler, 1986)

If R is a commutative ring spectrum of characteristic 2, then it is additively (and therefore multiplicatively) equivalent to HR.

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If we only require that identities are satisfied up to homotopy, then the rings we get are badly behaved.

If we keep track of the homotopies to ensure they behave in a coherent manner, then the rings are better-behaved.

A_n and E_n Structures

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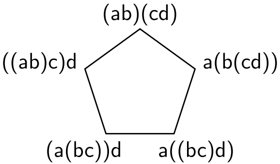
A choice of such an H is called an A_3 *structure* on R.

A_n and E_n Structures cont.

Using this homotopy, we can form a pentagon.



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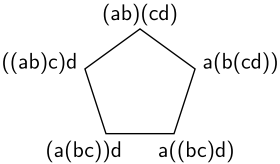

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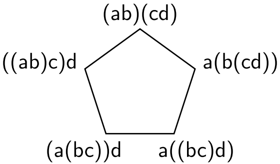

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A choice of nullhomotopy of the pentagon map (i.e. a cell that fills in the pentagon) is called an A_4 *structure* on R.

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Some bordism spectra, for example MO, MSO, MU, etc., admit natural E_∞ structures.

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There are E_n ring spectra of characteristic 2 which are not equivalent to the corresponding ring spectrum with the Boardman multiplication.

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There are *polynomial* E_n ring spectra $P_n(X)$ for any spectrum X. For $X = S^k$, the homology is freely generated by the Dyer-Lashof operations.

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Theorem (S.)

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$$HF_2 \wedge P_1(S^2) \wedge P_1(S^4) \wedge P_1(S^5) \wedge \cdots \to MO,$$

where the left-hand side has a smash factor $P_1(S^n)$ for every positive $n \neq 2^i - 1$.

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That is, as an E_1 or an E_2 ring spectrum, MO is 'polynomial'.

MO as an E_n ring spectrum cont.

This is not true in the E_3 case, as the homology of P_n is too big for $n \geq 3$.

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Modulo 2, it's actually a bordism invariant! Furthermore, it is additive (wrt. disjoint union) and multiplicative (wrt. cartesian product).

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It can be obtained by triangulating a space, then taking the alternating sum of the number of simplices used.

Alternatively, it is the alternating sum of the ranks of the homology groups of a space.

Modulo 2, it's actually a bordism invariant! Furthermore, it is additive (wrt. disjoint union) and multiplicative (wrt. cartesian product).

Therefore we have a ring homomorphism

$$\Omega_* \to \mathbb{F}_2[t],$$
$$M \mapsto \chi(M) t^{\dim M}.$$

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Unclear for E_n for $n \geq 3$ without knowing more about MO, or having a highly structured variant of Weber's method.

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