Higher algebra in characteristic 2

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Everything in the talk is implicitly derived. For example, (co)limits are really homotopy (co)limits when this makes sense, smash products are derived smash products, etc.

The bulk of this talk is really purloined from [Szy14].

1 The story, up to homotopy

Definition 1. A *graded Eilenberg–MacLane spectrum* (of characteristic 2), or *GEM* for short, is a spectrum of the form $\bigvee_{\mathcal{I}} \Sigma^i H\mathbb{F}_2$ for some indexing multiset $\mathcal{I}$.

Alternatively, you could think of these as the Eilenberg–MacLane spectra HR of graded $\mathbb{F}_2$-algebras R. These admit canonical ring structures up to homotopy, and can even be refined to E_∞ rings.

The homotopy theory of these objects is algebraic, as is covered in [Boa80].

Lemma 1 ([Boa80, Lemma 2.1])**.** *Let X be a GEM. Then the Hurewicz map $\pi_*X \to H_*X$ is injective, the embedded submodule given by the elements $x \in H_*X$ such that the dual Steenrod comodule action sends x to $1 \otimes x$.*

Lemma 2 ([Boa80, Lemma 2.3])**.** *Let X, Y be GEMs. Then (homotopy classes of) maps $f : X \to Y$ correspond to commutative squares*

$$\begin{array}{ccc} \pi_*X & \xrightarrow{f} & \pi_*Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ H_*X & \xrightarrow[f]{} & H_*Y \end{array}$$

where the top map is a module map, and the bottom is a map of $\mathcal{A}_*$-comodules.

This result can be used prove that there are no exotic homotopy ring spectrum structures on GEMs.

Lemma 3 ([Boa80, Theorem 1.1])**.** *Let X be a homotopy ring spectrum whose underlying object is a GEM. Then there is an isomorphism of ring spectra $X \simeq H\pi_*X$.*

It is feasible that there are homotopy ring spectra with $\pi_* X$ a graded $\mathbb{F}_2$-algebra but which are not additively GEMs. If we impose a commutativity condition, however, then this is not the case.

Theorem 1 ([PR85], [Wür86, Theorem 1.1])**.** *If X is a homotopy commutative ring spectrum of characteristic 2, then X is additively a GEM.*

Here, X having characteristic 2 simply means that $\pi_0 X$ has characteristic 2. Commutativity here is required – Wurgler gives some examples (e.g. spectra representing bordism with singularities) for which this doesn't hold.

2 (Weakly) initial E_n rings of characteristic 2

The initial rings Z/p of characteristic p are pretty fundamental, and it's natural to ask if there are analogues of these in higher algebra. The ring $\mathbb{Z}/2$ can be thought of as the cofiber $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$, and so the obvious candidate in our situation is the cofiber $S^0 \xrightarrow{\cdot 2} S^0 \to S^0/2$. This is given by the mod 2 Moore spectrum, but this does not admit a homotopy ring structure!

Another way to present $\mathbb{Z}/2$ is to take a pushout

$$\begin{array}{ccc} \mathbb{Z}[x] & \xrightarrow{x \mapsto 2} & \mathbb{Z} \\ {\scriptstyle 0 \mapsto x}\downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 \end{array}$$

in the category of commutative rings, which is given by the relative tensor product $\mathbb{Z} \otimes_{Z[x]} \mathbb{Z}$. In this context, we are guaranteed a ring structure.

To guarantee the existence of such objects, we work instead inside the category of E_∞-algebras.

Definition 2 ([Szy14, Definition 2.1])**.** The *versal E_∞ ring spectrum of characteristic 2* is given by the pushout

$$\begin{array}{ccc} \mathbb{P}(S^0) & \xrightarrow{\text{ev } 2} & \mathbb{S} \\ {\scriptstyle \text{ev } 0}\downarrow & \ulcorner & \downarrow \\ \mathbb{S} & \longrightarrow & \mathbb{S}/\!/2 \end{array}$$

in the category of E_∞ ring spectra.

Here, $\mathbb{P}(-)$ denotes the free functor Spectra $\to \mathrm{Alg}_{E_\infty}$(Spectra), and the maps $\mathrm{ev}(-)$ arise from the adjunction equivalence. This construction can be thought of as freely adding a nullhomotopy of 2 to the sphere, and so if an E_∞ ring X has characteristic 2 (i.e. has a nullhomotopy of 2), then it recieves a map from $\mathbb{S}/\!/2$.In fact, from its description as a pushout, we find

$$\begin{array}{ccc} \mathrm{Hom}_{E_\infty}(\mathbb{P}(S^0), X) & \longleftarrow & \mathrm{Hom}_{E_\infty}(\mathbb{S}, X) \\ \uparrow & & \uparrow \\ \mathrm{Hom}_{E_\infty}(\mathbb{S}, X) & \longleftarrow & \mathrm{Hom}_{E_\infty}(\mathbb{S}/\!/2, X) \end{array}$$

and using that $\mathrm{Hom}_{E_\infty}(\mathbb{S}, X) \simeq *$ because $\mathbb{S}$ is initial, and using the adjunction equivalence $\mathrm{Hom}_{E_\infty}(\mathbb{P}(S^0), X) \simeq \mathrm{Hom}_{\mathrm{Spectra}}(S^0, X) \simeq \Omega^\infty X$, the pullback is equivalent to

$$\begin{array}{ccc} \Omega^\infty X & \xleftarrow{\quad 2 \quad} & * \\ {\scriptstyle 0}\uparrow & & \uparrow \\ * & \longleftarrow & \mathrm{Hom}_{E_\infty}(\mathbb{S}/\!/2, X) \end{array}$$

so this is empty is 2 is not nullhomotopic, and otherwise is equivalent to $\Omega(\Omega^\infty X)$. Therefore maps out of these are not even unique up to homotopy if $\pi_1(X)$ is nontrivial.

This whole story admits a generalisation to E_n-algebras too, as developed in [ACB19]. We denote the corresponding E_n pushout by $\mathbb{S}/\!/_n 2$.

Equivalent methods of obtaining $\mathbb{S}/\!/_n 2$

In the E_∞ case, the pushout is a relative smash product, so there is an equivalence

$$\mathbb{S}/\!/2 \simeq \mathbb{S} \wedge_{\mathbb{P}(S^0)} \mathbb{S}$$

of E_∞ ring spectra. Note that the $\mathbb{P}(S^0)$-module structures on the left and right smash factors are different. In the E_n case, work of Hill–Lawson [HL21, Corollary 7.4] allows us to write some pushouts as relative smash products. This gives us an equivalence

$$\mathbb{S}/\!/_n 2 \simeq \mathbb{S} \wedge_{\mathbb{P}_{n+1}(S^0)} \mathbb{S}.$$

These can also be obtained as Thom spectra of the map $\Omega^n \Sigma^n S^1 \to BO$ induced by the non-trivial homotopy class $S^1 \to BO$. This gives us the homology

$$H_* \mathbb{S}/\!/_n 2 \cong H_* \Omega^n \Sigma^n S^1 \cong \mathbb{F}_2[a, Q_1^{i_1} \cdots Q_{n-1}^{i_{n-1}} a],$$

which had previously been computed by Cohen.

Since all these things are homotopy commutative and have characteristic 2, you can determine the homotopy by 'dividing by' the dual Steenrod algebra. Since

$$H_* \mathbb{S}/\!/_2 2 \cong \mathcal{A}_*,$$

this means $\pi_* \mathbb{S}/\!/_2 2 \cong \mathbb{F}_2$, meaning $\mathbb{S}/\!/_2 2 \simeq H\mathbb{F}_2$.

3 Applications to E_n maps

Because $H\mathbb{F}_2$ is E_∞ and has homotopy concentrated in degree 0, there is an essentially unique map $\mathbb{S}//2 \to H\mathbb{F}_2$.

Theorem 2 ([Szy14, Proposition 5.1])**.** *There is no E_∞ map $H\mathbb{F}_2 \to \mathbb{S}//2$.*

Proof. Any such map postcomposes with $\mathbb{S}//2 \to H\mathbb{F}_2$ to give the identity (up to homotopy) by our classification of E_n endomorphisms of versal algebras, so sends ξ_1 to a. Furthermore, the map must commute with the Dyer–Lashof operations, predicting that $Q_2 a = Q_2 f(\xi_1) = f(Q_2\xi_1) = f(\xi^4) = a^4$. Since this relation does not hold in $\mathbb{S}//2$, we derive a contradiction. □

Definition 3. The *unoriented bordism spectrum*, denoted MO, is the Thom spectrum of the delooped j-homomorphism $BO \to B\mathrm{GL}_1\mathbb{S}$.

A result of Thom's original paper is that $\pi_* MO$ is the unoriented bordism ring, permitting him to compute that

$$\pi_* MO \cong \mathbb{F}_2[x_i : i \neq 2^j - 1].$$

Knowing that MO is E_∞ and that the bordism ring has characteristic 2 allows us to use the results of the first part to show that MO is a GEM.

If this is the whole story, then one might expect that there is an E_∞ equivalence $H\pi_* MO \simeq MO$. This is not the case!

Theorem 3 (Helen Gilmour's Thesis)**.** *There is no E_4 map $H\mathbb{F}_2 \to MO$.*

The existence of a map is obstructed by a Dyer–Lashof operation (it must be a Q_3). These were computed in $H\mathbb{F}_2$ in the H_∞ book, and in BO (and therefore MO) by Priddy/Kochman.

To try and recover from this, one might expect that if $\pi_* MO$ is a polynomial ring, MO might be 'polynomial' over $\mathbb{S}//2$, but this is not the case!

Theorem 4 ([Szy14, Corollary 5.11])**.** *There is no spectrum X such that there exists an equivalence*

$$MO \simeq \mathbb{S}//2 \wedge \mathbb{P}(X)$$

of E_∞ ring spectra.

Proof. Otherwise there would be a retraction

$$\mathbb{S}//2 \to MO \simeq \mathbb{S}//2 \wedge \mathbb{P}(X) \to \mathbb{S}//2 \wedge \mathbb{P}(*) \simeq \mathbb{S}//2,$$

but $H_* MO$ is too small to support this. Actually, we can use another Dyer–Lashof operation argument: the image of $Q_2 a$ is x_1^4, so the map $\mathbb{S}//2 \to MO$ is not a homology injection. □

This raises the question – what does MO look like as an E_∞ ring spectrum?

4 Highly structured models of MO

It turns out that the combinatorics of H_*BO as an algebra over the E_2 Dyer–Lashof operations is sufficient to determine that

$$MO \simeq_{E_1} H\mathbb{F}_2 \wedge \bigwedge_{i \neq 2^j - 1} \mathbb{P}_1(S^i),$$

and

$$MO \simeq_{E_2} H\mathbb{F}_2 \wedge \bigwedge_{i \in \mathbb{Z}_{>0}} \mathbb{P}_2(S^{2i}),$$

so Thom's picture is accurate on the level of E_1 and E_2 ring spectra. Note that the smash product is not the coproduct of E_n rings for $n \neq \infty$.

We know (from the earlier result about the non-existence of an E_4 map $H\mathbb{F}_2 \to MO$) that this won't be the case for E_4, but what about E_3?

A quick aside on H_∞ and E_∞

The calculations I'm going to talk about only involve the H_n structure of MO (the Dyer–Lashof operations), but turn out to give E_n models of MO. Noel and Lawson both construct instances of H_∞ ring spectra that do not refine to E_∞ ring spectra, and Johnson–Noel construct an obstruction theory that deals with this. Computing the obstruction groups is hard. It's possible that in our cases of interest, the obstruction theory allows for our H_∞ maps to be refined in a unique or canonical way, but the calculations I am going to talk about are somehow elementary in comparison.

There are no obstructions to the existence of an E_3 map $H\mathbb{F}_2 \to MO$ on the level of homology, but it remains to try and construct one. If we can do this, the combinatorics of H_*BO immediately give an equivalence

$$MO \simeq_{E_3} H\mathbb{F}_2 \wedge \mathbb{P}_3(S^2) \wedge \bigwedge_{i \in \mathbb{Z}_{>0}} \mathbb{P}_3(S^{4i}).$$

An E_3 model of $H\mathbb{F}_2$

I still have things to check here, so take this last part of the talk with a pinch of salt.

Our aim is to equip $H\mathbb{F}_2$ with an E_3 universal property, so that construction of an E_3 map $H\mathbb{F}_2 \to MO$ becomes trivial. The ideal thing would be to approximate $H\mathbb{F}_2$ using E_3 pushouts, iteratively adding and killing classes in the relevant degrees to get successively better approximations. However, computing non-E_3 invariants of E_3 pushouts is difficult. Results of Hill–Lawson from earlier, however, tell us that some E_3 pushouts are equivalent to relative smash products, and invariants of these can be computed using the Kunneth spectral sequence.

The pushouts that can be computed as relative smash products come from *actions*, classified by homotopy classes $S^i \to Z_n(X)$, where Z_n denotes the E_n

centre of X. We can iteratively construct the relevant homotopy classes using universal properties of more structured variants of X.

References

[ACB19] Omar Antolín-Camarena and Tobias Barthel. A simple universal property of Thom ring spectra. *J. Topol.*, 12(1):56–78, 2019.

[Boa80] J. M. Boardman. Graded Eilenberg-MacLane ring spectra. *Am. J. Math.*, 102:979–1010, 1980.

[HL21] Michael A. Hill and Tyler Lawson. E_k-pushouts and E_{k+1}-tensors. Preprint, arXiv:2110.07429 [math.AT] (2021), 2021.

[PR85] A. V. Pazhitnov and Yu. B. Rudyak. On commutative ring spectra of characteristic 2. *Math. USSR, Sb.*, 52:471–479, 1985.

[Szy14] Markus Szymik. Commutative $\mathbb{S}$-algebras of prime characteristics and applications to unoriented bordism. *Algebr. Geom. Topol.*, 14(6):3717–3743, 2014.

[Wür86] Urs Würgler. Commutative ring-spectra of characteristic 2. *Comment. Math. Helv.*, 61:33–45, 1986.